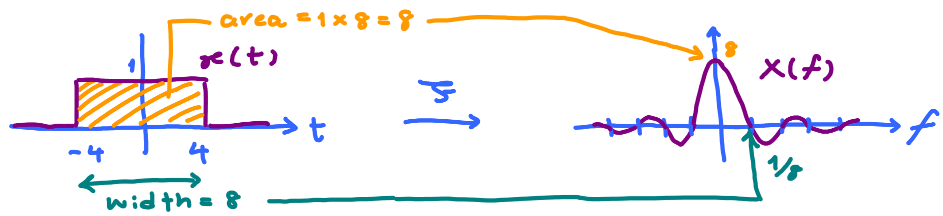
Q1 Rectangular pulses and sinc functions

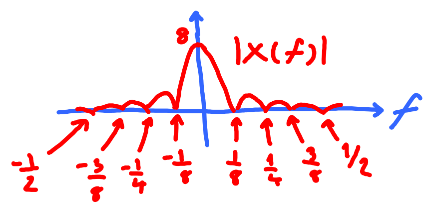
Wednesday, August 26, 2015 8:36 PM

All of the signals under consideration here are rectangular pulses in the time domain. We know that the Fourier transform of an even rectangular pulse is a sinc function in the freq. domain.

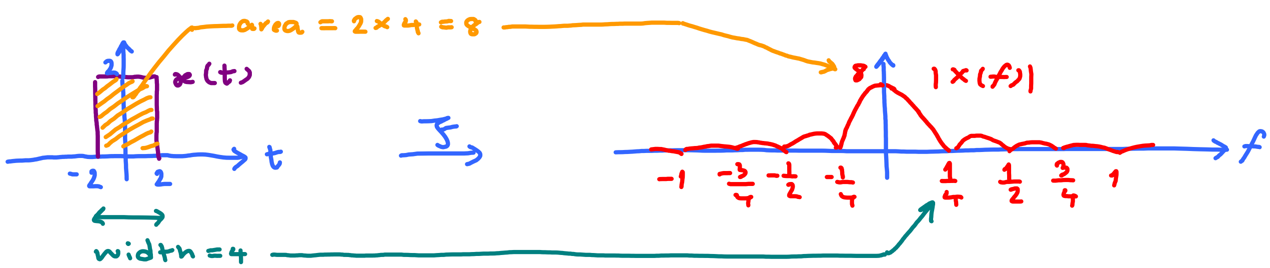
(a)



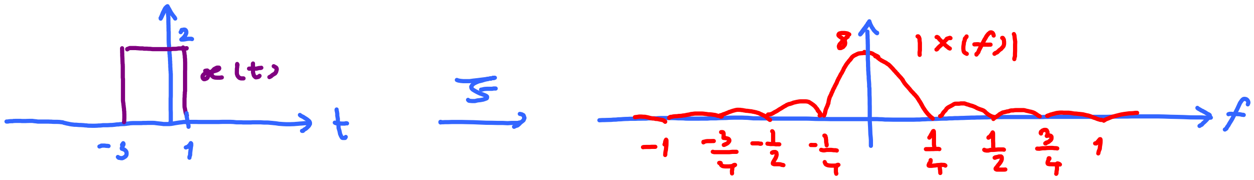
Note that we are not done yet. We are asked to plot the amplitude spectrum ($|X(f)|$) and not the Fourier transform $X(f)$ itself. Because $X(f)$ is real-valued, $|X(f)|$ is simply the absolute value of $X(f)$. So, the "negative" part of $X(f)$ gets rectified (flipped up):



(b) We follow the same technique that was applied in part (a).



(c) Note that the signal $x(t)$ in this part is the same as the one in part (b) but shifted to the left. We have seen in class that time-shifting of the whole signal does not change its amplitude spectrum. Therefore, we can simply copy the plot of $|X(f)|$ from part (b) here.



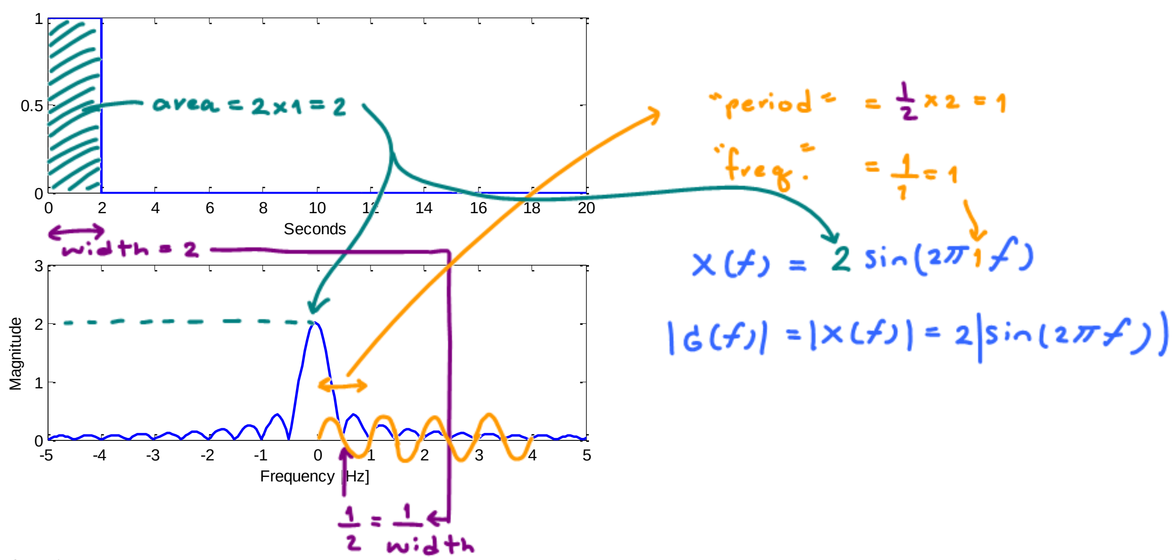
Q2 Magnitude Spectrum via MATLAB

a)

i) The plot in the time domain shows a rectangular pulse $g(t)$ on the interval [0, 2]. In class, we have seen how to find the Fourier transform of rectangular functions that are even functions (symmetric wrt. the vertical axis). Our $g(t)$ is not. We need to shift it to the left by 1 to get a rectangular pulse $1[|t| \le 1]$ which is even.

As we have discussed in class, time shifting does not change the amplitude spectrum. Hence, $|G(f)|$ is the <u>same as the magnitude of</u> the Fourier transform of $x(t) \equiv 1[|t| \le 1]$.

Method 1:



Method 2:

In class, we have seen an example of a Fourier transform pair where we have rectangular pulse of width T_0 centered at origin in the time domain. In particular, we know that

$$1\left[|t| \le \frac{T_0}{2}\right] \xrightleftharpoons{\mathcal{F}} T_0 \operatorname{sinc}(\pi T_0 f).$$

Pluggging in the value of 2 for the width, we have

$$1[|t| \le 1] \xrightleftharpoons{\mathcal{F}} 2\operatorname{sinc}(2\pi f).$$

Therefore,

$$|G(f)| = 2|\operatorname{sinc}(2\pi f)|.$$

ii) In the bottom part of Figure (i) below, the theoretical expression in part (i) is plotted using the "x" marks on top of the provided plot from `specrect.m`. The marks match the theoretical plot. Therefore, the expression above agrees with the plot via MATLAB's `plotspect.m`.

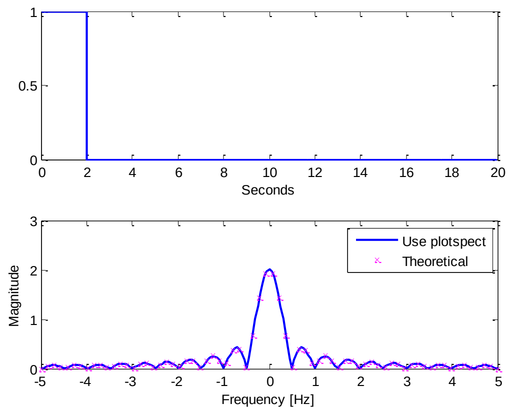


Figure (i)

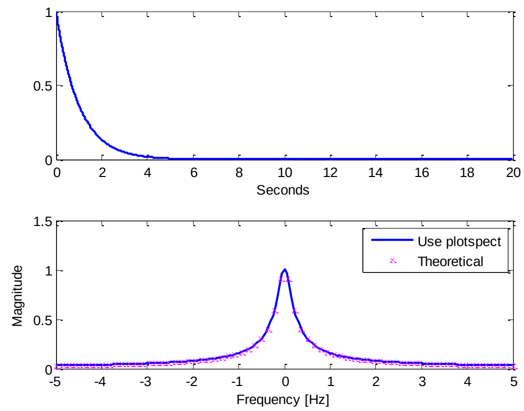


Figure (ii)

b)

i) See Figure (ii) above.

ii) By the Fourier-transform formula,

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(1+j2\pi f)t}dt$$

$$= \frac{1}{-(1+j2\pi f)} e^{-(1+j2\pi f)t}\Big|_{t=0}^{\infty} = \frac{1}{1+j2\pi f}$$

Recall that the magnitude of a complex number $z = x + jy$ is $|z| = \sqrt{x^2 + y^2}$ and that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

Therefore, $|S(f)| = \frac{1}{\sqrt{1+(2\pi f)^2}}$.

iii) The $|S(f)|$ derived analytically is plotted in Figure (ii) using the "x" marks on top of the plots from `plotspect.m`. They are virtually identical.

iv) Here is the result displayed on the command window:

```
>> SymbFourier
S =
1/(a + pi*f*2*i)
```

With variable "a" in the m-file set to 1, we have same result as in (i).

Q3 Sinc Function and Triangular Signal

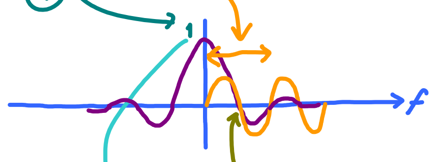
Wednesday, July 06, 2011 12:16 PM

(a) We will try to solve this problem graphically.

(i) To do this, we first plot $X(f) = \mathrm{sinc}(5\pi f) \overset{①}{=} \frac{\sin(2\pi \frac{5}{2} f)}{5\pi f}$.

"period" $= \frac{2}{5}$ ③ $\Leftarrow$ "freq" $= \frac{5}{2}$ ②

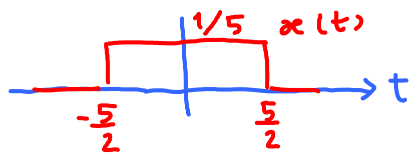
⑤



④ $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$

So, the rectangular pulse in the time domain has width = 5.

Its area under the graph must be 1. So, its height must be $\frac{1}{5}$.



(ii) $\displaystyle\int_{-\infty}^{\infty} X(f)\,df = x(0) = \frac{1}{5}$.

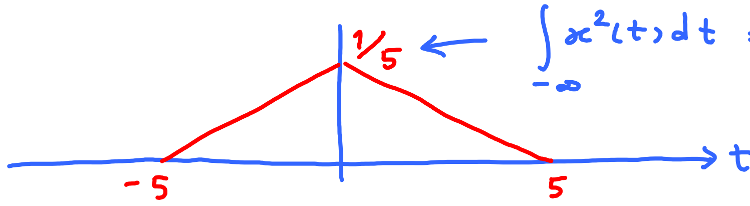
(b) Note that $Y(f) = (X(f))^2 = X(f) \times X(f)$. ← multiplication

By the convolution property of Fourier transform, we know that

$y(t) = x(t) * x(t)$ ← we discussed this convolution of rect. functions in class.

↑ convolution.

(i)



$\displaystyle\int_{-\infty}^{\infty} x^2(t)\,dt = \left(\frac{1}{5}\right)^2 \times 5 = \frac{1}{5}$

(ii) $\displaystyle\int_{-\infty}^{\infty} Y(f)\,df = y(0) = \frac{1}{5}$.

Alternatively, one can use the Parseval's theorem:

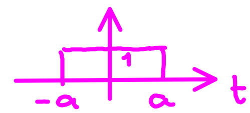
$$\int_{-\infty}^{\infty} Y(f)\,df = \int_{-\infty}^{\infty} X^2(f)\,df = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{5}.$$

$\int_{-\infty}$ $\int_{-\infty}$ $-\infty$

Note that we don't have to write $|\cdot|$ because $x(t)$ and $X(f)$ are real-valued.

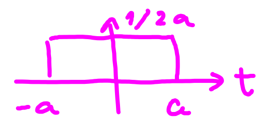
Alternatively, we can try to solve (a.i) and (b.i) via formula.

(a.i) We know that



$$2a \operatorname{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} 1[|t| \le a]$$ ← shown in class.

Therefore, $\operatorname{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a]$.



Here, $2a = 5$. So, $a = 5/2$.

(b.i)

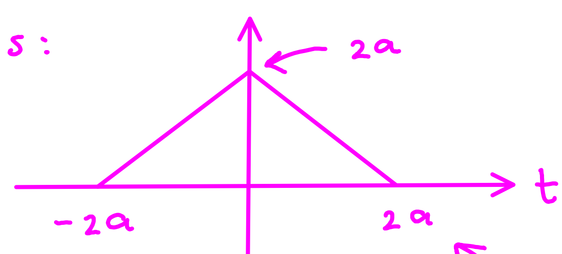
$$\operatorname{sinc}^2(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a] * \frac{1}{2a} 1[|t| \le a]$$

$$= \frac{1}{4a^2}\left(1[|t| \le a] * 1[|t| \le a]\right)$$

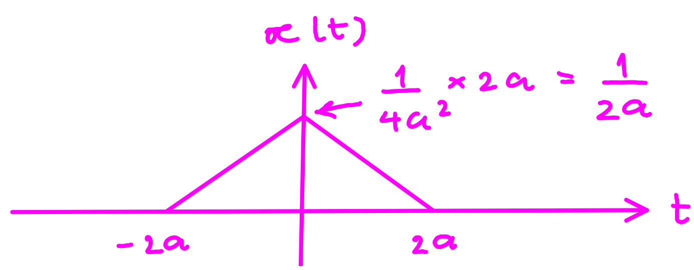
So, we can solve this question if we can find the convolution of $1[|t| \le a]$ with itself.

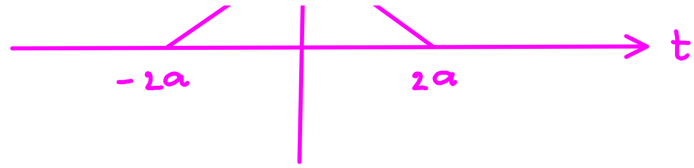
This is also discussed in class:

$$1[|t| \le a] * 1[|t| \le a] =$$



Therefore, the plot of $x(t)$ should be the same as but scaled vertically by a factor of $1/4a^2$:




t
-2a
2a

Q4 Using Properties of FT

Thursday, August 27, 2015 8:54 PM

(b) Note that $g_1(t) = g(-t)$.

Recall that $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$.

Here, $a = -1$

Therefore, $G_1(f) = \frac{1}{|-1|} G\left(\frac{f}{-1}\right) = \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)$

(c) Note that $g_2(t) = g(t-1) + g_1(t-1)$

$$\Rightarrow \quad G_2(f) = e^{-j2\pi f} G(f) + e^{-j2\pi f} G_1(f)$$

Here, we write "ω" instead of "$2\pi f$".
After we're done massaging the expressions, we change "ω" back to "$2\pi f$"

$$= \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1 + e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(2\cos(\omega) - j\omega(2j)\sin\omega - 2\right)$$

$$= \frac{2e^{-j2\pi f}}{(2\pi f)^2}\left(\cos(2\pi f) + 2\pi f \sin(2\pi f) - 1\right)$$

(d) Note that $g_3(t) = g(t-1) + g_1(t+1)$

$$\Rightarrow \quad G_3(f) = e^{-j2\pi f} G(f) + e^{j2\pi f} G_1(f)$$

$$= \frac{1}{\omega^2}\left(1 - \cancel{j\omega} - e^{-j\omega} + 1 + \cancel{j\omega} - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}\left(2 - e^{-j\omega} - e^{j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\omega} - 2 + e^{-j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)^2$$

$$= -\frac{1}{\omega^2}\left(2j\sin\frac{\omega}{2}\right)^2 = \frac{\sin^2\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2} = \text{sinc}^2\left(\frac{\omega}{2}\right) = \text{sinc}^2(\pi f)$$

(e) Note that $g_4(t) = g\left(t - \frac{1}{2}\right) + g_1\left(t + \frac{1}{2}\right)$.

use part (b)

$$\Rightarrow \quad G_4(f) = e^{-j\omega/2} G(f) + e^{j\omega/2} G_1(f)$$

$$= e^{-j\omega/2}\frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + e^{j\omega/2}\frac{1}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{1}{\omega^2}\left(\cancel{e^{j\frac{\omega}{2}}} - j\omega e^{j\frac{\omega}{2}} - \cancel{e^{-j\frac{\omega}{2}}} + \cancel{e^{-j\frac{\omega}{2}}} + j\omega e^{-j\frac{\omega}{2}} - \cancel{e^{j\frac{\omega}{2}}}\right) = \frac{-j}{\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= \frac{(-j)}{\omega}(2j)\sin(\omega/2) = \frac{\sin(\omega/2)}{\omega/2} = \text{sinc}\left(\frac{\omega}{2}\right) = \text{sinc}(\pi f)$$

(f) Note that $g_5(t) = 1.5\, g\left(\frac{1}{2}(t-2)\right)$

$$\Rightarrow \quad G_5(f) = 1.5 \times \frac{1}{1/2}\, G\left(\frac{f}{1/2}\right) e^{-j2\omega} = 3\, G(2f)\, e^{-j}$$

$$= 3 \times \frac{1}{(2\pi 2f)^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right)e^{-j2\omega} = \frac{3}{4\omega^2}\left(1 - 2j\omega - e^{-2j\omega}\right)$$

$$= \frac{3}{4(2\pi f)^2}\left(1 - j4\pi f - e^{-j4\pi f}\right)$$